Electoral Equilibria under Scoring Voting Rules

Dodge Cahan, John McCabe-Dansted and Arkadii Slinko

Department of Mathematics
The University of Auckland

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The aim of the paper

The aim is to generalise the Hotelling-Downs model to explain the so-called ‘Principle of Local Clustering’.

We will do this by using more general scoring rules instead of Plurality.

Principle of Local Clustering (Eaton-Lipsey, 1975)

“When a new firm enters a market, or when an existing firm relocates, there is a strong tendency for that firm to locate as close as possible to another firm. This behaviour tends to create local clusters of firms in many equilibrium and disequilibrium situations.”

Introduction

Parties running in an election for a parliament must decide where they stand on the ideological spectrum in order to maximise the support of the voters measured by some voting rule.

Firms on the Main Street have to decide where to locate to get a larger share of the customers.

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- Do equilibrium situations exist?
- What kind of equilibria?

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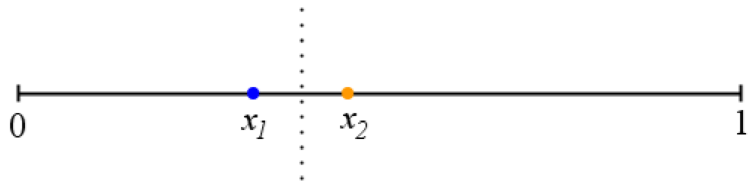

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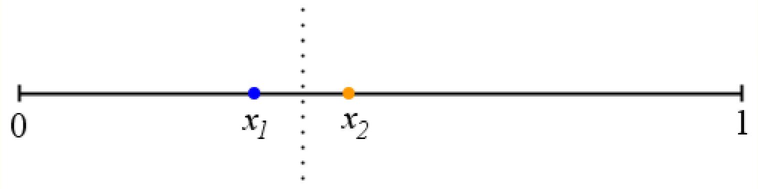

- There are m candidates. A *profile* is an m vector $x = (x_1, \ldots, x_m) \in [0,1]^m$ that specifies each candidate's position: x_i is candidate i's position.

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- The basic theme of Myerson's Schumpeter Lecture (1998, Berlin meetings of the European Economic Association) is the importance of explicitly comparing different electoral systems in Hotelling type models.
- Myerson concentrated on positional scoring rules, we follow him in this.

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- Candidates are *score maximisers*.

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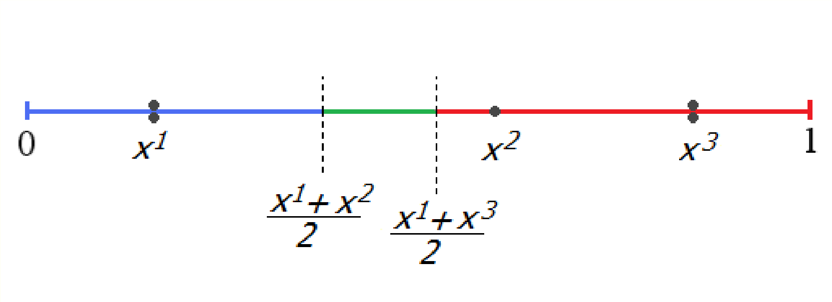
- For example, if Borda rule is used:

Ranking	Points received
A	6
B	5
$C \sim D \sim E$	$3 = \frac{1}{3}(4 + 3 + 2)$
F	1
G	0

Workings of a positional scoring rule



The score of a candidate positioned at x^1 would be

$$\frac{s_1+s_2}{2}\frac{x_1+x_2}{2}+\frac{s_2+s_3}{2}\frac{x_3-x_2}{2}+\frac{s_4+s_5}{2}\left(1-\frac{x_1+x_3}{2}\right).$$

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- The issue space is a road through an urban area, with customers distributed along it.
- Instead of candidates, we have firms seeking to maximise their share of the market.
- Firms are not allowed to compete on price.
- The scoring rule $s = (s_1, \ldots, s_m)$ is a vector of probabilities, s_i being the probability that a customer buys from i-th nearest firm.

Nash equilibrium

- We look for profiles (vectors of candidate positions) that are in *Nash equilibrium*.
- This is a situation in which no candidate has an incentive to change position. Each candidate's position is a best response to positions of the other candidates.
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- A *convergent* Nash equilibrium (CNE) occurs when all candidates adopt the same ideological position.
- A *non-convergent* Nash equilibrium (NCNE) is when not all candidate positions are the same.

Question

- In Plurality no Nash equilibrium may have more than two firms at the same location.
- For $m > 2$ Plurality rule cannot explain the tendency of firms to cluster together at different locations.
- Can we explain the clustering tendency by a more general model using scoring rules?

Convergent equilibria

Theorem (Cox, 1987). For m candidates and scoring rule s, a profile $x = (x^*, \ldots, x^*)$ is a CNE if and only if

$$c(s,m) \leq x^* \leq 1 - c(s,m), \qquad (1)$$

where $c(s,m) = \dfrac{s_1 - \bar{s}}{s_1 - s_m}$ is the *c-value* (with $\bar{s} = \frac{1}{m}\sum_{i=1}^{m} s_i$) .

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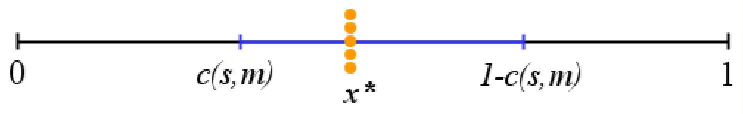

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- It is an easy observation that in a three-candidate election under any positional scoring rule no NCNE exist.
- The first question: If $m = 4$, can we characterize the rules for which NCNE exist?

The four-candidate case

Theorem (CMS., 2012). In a four-candidate election under scoring rule $s = (s_1, s_2, s_3, s_4)$, NCNE exist iff both the following conditions are satisfied:

- a) $c(s, 4) > 1/2$ (that is no CNE exist);
- b) $s_1 > s_2 = s_3$.

Moreover, the NCNE is unique and symmetric. Two paired candidates at $x_1 = \dfrac{s_1}{4(s_1 - s_2)}$ and $x_2 = 1 - x_1$.

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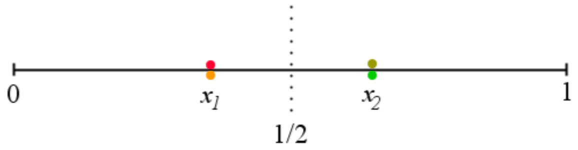

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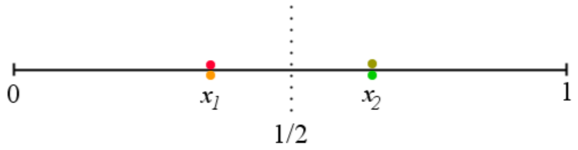

If $c(s, 4) > 1/2$ but $s_2 \neq s_3$ then no NE of either kind exist.

The five-candidate case

Theorem (CMS., 2012). In a five-candidate election under scoring rule $s = (s_1, s_2, s_3, s_4, s_5)$, NCNE exist iff both the following conditions are satisfied:

- a) $s_1 > s_2 = s_3 = s_4$;
- b) $c(s, 5) > 1/2$.

Moreover, the NCNE is unique and symmetric, with equilibrium profile $x = ((x^1, 2), (1/2, 1), (x^2, 2))$, where

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Note. For both $m = 4$ and $m = 5$ CNE and NCNE cannot coexist together. This will be broken for $m = 6$.

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Theorem (CMS., 2012). Given $m = 6$ and scoring rule $s = (s_1, s_2, s_3, s_4, s_5, s_6)$. Then there are four possible types of equilibria split in two groups:

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Convex scores

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for some $1 \leq n < m$. Then there are no NCNE, unless the subrule $s' = (s_1, \ldots, s_n, s_{n+1})$ is Borda and $n + 1 \leq \lfloor m/2 \rfloor$ (i.e., more than half the scores are constant). In the latter case NCNE do exist.

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Example. $s = (3, 2, 1, 0, 0, 0, 0)$.

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That is, for every drop at the top end there is a drop at least as large at symmetric position at the bottom end.

Concave and weakly concave scores

We say that the rule $s = (s_1, \ldots, s_m)$ is concave if

$$s_1 - s_2 \leq s_2 - s_3 \leq \ldots \leq s_{m-1} - s_m.$$

Most our positive results are, however, applicable to a larger class of rules.

We say that a scoring rule is weakly concave if it obeys the following property:

$$s_i - s_{i+1} \leq s_{m-i} - s_{m-i+1},$$

for all $1 \leq i \leq \lfloor m/2 \rfloor$.

That is, for every drop at the top end there is a drop at least as large at symmetric position at the bottom end.

A weakly concave rule is either worst-punishing or intermediate.

Surprising properties of weakly convex rules

Theorem (CMS, 2012). Any weakly concave scoring rule s has no NCNE

$$x = ((x^1, n_1), \ldots, (x^q, n_q))$$

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This means that if a concave rule has an NCNE it has to have more than half of all candidates in one of the extreme locations!

Such weakly convex rules exist

For $m = 12$ the scoring rule $s = (4, 4, 4, 3, 3, 3, 2, 1, 1, 0, 0, 0)$ satisfies weak concavity, yet does allow NCNE. In particular, the profile

$$((x^1, n_1), (x^2, n_2)) = \left(\left(\frac{13}{28}, 8 \right), \left(\frac{41}{84}, 4 \right) \right)$$

with eight candidates at position $x^1 = \frac{13}{28}$ and four at position $x^2 = \frac{41}{84}$ is an NCNE.

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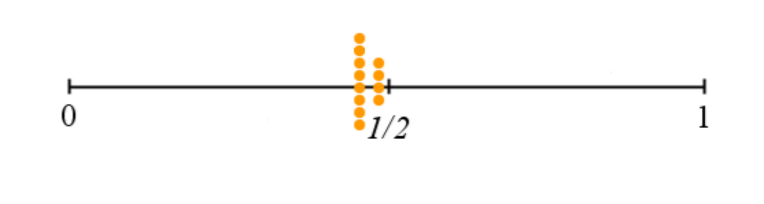

Rules that allow clustered NCNE

Theorem. Let there be $m = qr$ candidates, $q \geq 2$, and

$$s = (s_1, \ldots, s_{r-1}, 0, \mid \underbrace{0, \ldots, 0}_{r} \mid \cdots \mid \underbrace{0, \ldots, 0}_{r})$$

be a scoring rule where only the first $r - 1$ scores are nonzero. Let $s' = (s_1, \ldots, s_{r-1}, 0)$ be its r-subrule.

Divide the interval into q equally sized subintervals. Then the profile in which r candidates locate at the half-way point of each subinterval is a NCNE if and only if $c(s', r) \leq 1/2$.

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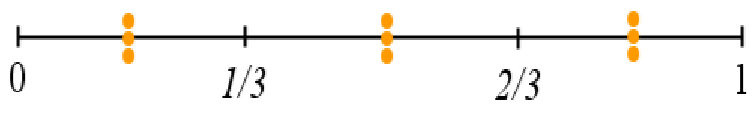

Example: $m = 9$ candidates and $q = r = 3$. This is a NCNE for rules: 2-approval, $s = (1, 1, 0, 0, 0, 0, 0, 0, 0)$ or $s = (2, 1, 0, 0, 0, 0, 0, 0, 0)$.

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- If consumers predominately buy from the nearst firm we get clustered NCNE – firms congregate at multiple locations spread through the city.
- If customers often buy from distant firms, we get CNE — all firms congregate in the central business district.

Conclusions

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- When the most important thing is to place first, candidates are better off adopting more extreme positions.
- In NCNE candidates spread along the issue space grouped into clusters.
- Plurality, $s = (1, 0, \ldots, 0)$ does not explain the Principle of Local Clustering but more general scoring rules do.

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Thanks for your attention!